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Matching Dominating Sets of Strong Product Graph of Euler Totient Cayley Graphs with Arithmetic V_n Graphs

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Abstract— Theory of domination in graphs introduced by Ore and Berge is an emerging area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology.

Products are often viewed as a convenient language with which one can describe structures, but they are increasingly being applied in more substantial ways. Every branch of mathematics employs some notion of product that enables the combination or decomposition of its elemental structures.

In this paper, we consider strong product graph of Euler totient Cayley graphs with Arithmetic V_n graphs and present some results on matching domination parameter of these graphs.

Index Terms— Arithmetic V_n graph, Matching dominating set, Euler totient Cayley graph, Strong product graph.

Subject Classification: 68R10

1 Introduction

Graph Theory has been realized as one of the most useful branches of Mathematics of recent origin with wide applications to combinatorial problems and to classical algebraic problems. Graph theory has applications in diverse areas such as social sciences, linguistics, physical sciences, communication engineering etc.

The theory of domination in graphs is an emerging area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology.

The strong product was studied by Nesetril [1]. Occasionally it is also called as strong direct product or symmetric composition. This product is commutative and associative as an operation on isomorphism classes of graphs [2].

Strong product graph of $G(Z_n,\varphi)$ with $G(V_n)$

In this paper we consider the strong product graph of Euler totient Cayley graph with Arithmetic V_n graph. The properties of strong product graph are studied by Uma Maheswari [3].

Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, \dots, u_l\}$

and $V_2 = \{v_1, v_2, \dots, v_m\}$ respectively. Then the strong product of these two graphs denoted by $G_1 \boxtimes G_2$ is defined as a graph whose vertex set is $V_1 \times V_2$ and any two distinct vertices (u_1,v_1) and (u_2,v_2) of $G_1 \boxtimes G_2$ are adjacent if

(i) $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or

(ii) $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or

(iii) $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

Let G_1 denote Euler totient Cayley graph graph and G_2 denote Arithmetic V_n graph. Then G_1 and G_2 have no loops and multiple edges. Hence by the definition of strong product, $G_1 \boxtimes G_2$ is also a simple graph.The strong product $G_1 \boxtimes G_2$ is a complete graph, if n is a prime.

2 Euler Totient Cayley Graph $G(Z_n,\varphi)$

For any positive integer n, let Z_n be the additive group of integers modulo n and let S be the set of all numbers less than n and relatively prime to n. That is $S = \{r/1 \le r < n$ and $GCD(r,n) = 1\}$. Then $|S| = \varphi(n)$, where φ is the Euler totient function. We can see that S is a symmetric subset of the group $(Z_n, \oplus)$.

The Euler totient Cayley graph $G(Z_n,\varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2,\dots,n-1\}$ and the edge set is $E = \{(x,y)/x - y \in S$ or $y - x \in S\}$. This graph is denoted by $G(Z_n,\varphi)$.

The matching domination parameter of these graphs is studied by the authors [4] and we require the following results and we present them without proofs.

Theorem 2.1: If $n = p$, then matching domination number of $G(Z_n,\varphi)$ is 2.

Theorem 2.2: If $n = 2p$, where p is an odd prime, then matching domination number of $G(Z_n,\varphi)$ is 4.

Theorem2.3: Let n be neither a prime nor $2p$ and

$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$, where $p_1, p_2, p_3, \dots, p_k$ are

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primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are integers ≥ 1.Then the matching domination number of $G(Z_n, \varphi)$ is given by

$$\gamma_m(G(Z_n, \varphi)) = \begin{cases} \lambda + 1 & \text{if } \lambda \text{ is odd,} \\ \lambda + 2 & \text{if } \lambda \text{ is even.} \end{cases}$$

where λ is the length of the longest stretch of consecutive integers in V each of which shares a prime factor with n.

3 Arithmetic V_n Graph

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u and v are adjacent in V_n graph if and only if $GCD(u, v) = p_i$ for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex.

In this chapter we made an attempt to study some domination parameters of these graphs. In doing so we have deleted the vertex 1 from the graph as the contribution of this isolated vertex is nothing, when domination parameters are enumerated.

Clearly, V_n graph is a connected graph.

The matching domination parameter of these graphs is studied by the authors [5] and we require the following results and we present them without proofs.

Theorem 3.1: If n is a prime, then matching domination number does not exist for the graph $G(V_n)$.

Theorem 3.2: If $n \neq p$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots\dots p_k^{\alpha_k}$, where $\alpha_i \geq 1$,then the matching domination number of $G(V_n)$ is given by

$$\gamma_m(G(V_n)) = \begin{cases} k & \text{if k is even,} \\ k+1 & \text{if k is odd.} \end{cases}$$

where k is the core of n.

4 Matching Domination in Strong product graph

In this section we find minimum matching dominating sets of strong product graph of $G(Z_n, \varphi)$ with $G(V_n)$ graph and obtain its matching domination number in various cases.

Perfect Matching

A matching M in G is called a perfect matching if every vertex of G is incident to some edge of M.

Matching Domination

A dominating set D of a graph G is said to be a matching dominating set if the induced subgraph $\langle D \rangle$ admits a perfect matching.

The minimum cardinality of a matching dominating set is called the matching domination number of G and is denoted by $\gamma_m(G)$.

Theorem 4.1: If n is a prime, then the matching domination number of $G_1 \boxtimes G_2$ is 2.

Proof: Let n be a prime. Then the graph $G_1 \boxtimes G_2$ is a complete graph and hence every vertex is of degree $n-1$. So, any single vertex dominates all other vertices in $G_1 \boxtimes G_2$. Let $D = \{(0, p)\}$, where 0 is a vertex in G_1 and p is a vertex in G_2. Let V denote the vertex set of $G_1 \boxtimes G_2$.

Then every vertex in $V - D$ is adjacent to the vertex $\{(0, p)\}$ in D.

Therefore $\gamma(G_1 \boxtimes G_2) = 1$.

For any $t \in \{1, 2, \dots\dots, n-1\}$ in G_1 the vertex $(0, p)$ is adjacent to the vertex (t, p), since the graph $G_1 \boxtimes G_2$ is a complete graph.

So, if $D = \{(0, p), (t, p)\}$, then the induced subgraph $< D >$ admits a perfect matching with minimum cardinality.

Hence D is a minimal matching dominating set of $G_1 \boxtimes G_2$.

Therefore $\gamma_m(G_1 \boxtimes G_2) = 2$.

Theorem 4.2: Let $n \neq p$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} p_2^{\alpha_2} \dots\dots p_k^{\alpha_k}$, where $\alpha_i \geq 1$. Then the matching domination number of $G_1 \boxtimes G_2$ is given by

$$\gamma_m(G_1 \boxtimes G_2) = \begin{cases} (\lambda + 1) \cdot (k-1) & \text{if } \alpha_i = 1 \text{ for more than one i,} \\ (\lambda + 1) \cdot k & \text{otherwise.} \end{cases}$$

if λ is odd and

$$\gamma_m(G_1 \boxtimes G_2) = \begin{cases} (\lambda + 2) \cdot (k-1) & \text{if } \alpha_i = 1 \text{ for more than one i,} \\ (\lambda + 2) \cdot k & \text{otherwise.} \end{cases}$$

if λ is even, where k is the core of n and λ is the length of the longest stretch of consecutive integers in V_1 of G_1 each of which shares a prime factor with n.

Proof: Suppose n is not a prime and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots p_k^{\alpha_k}$ where $\alpha_i \geq 1$.

Consider the graph $G_1 \boxtimes G_2$ Let V_1, V_2, V denote the vertex sets of G_1, G_2 and $G_1 \times G_2$ respectively.

By Theorem 2.3, we know that $D_1 = \{u_{d_1}, u_{d_2}, \dots, u_{d_{\lambda+1}}\}$ is a dominating set of G_1 with cardinality $\lambda + 1$, where $\neq 2p$. But it can be easily seen that this set also becomes a dominating set even in the case $n = 2p$.

The following cases arise.

Case 1: Suppose $\alpha_i > 1$ for all i or $\alpha_i = 1$ for only one i.

We know that $D_1 = \{u_{d_1}, u_{d_2}, \dots, u_{d_{\lambda+1}}\}$ is a dominating set of G_1. As in the Remark of Theorem 3.6.4 of Chapter 3 [6], we know that $D_2 = \{p_1, p_2, \dots\dots, p_k\}$ is a dominating set of G_2 with cardinality k.

Let $D = D_1 \times D_2 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \times \{p_1, p_2, \ldots, p_k\}$. We now show that D is a dominating set of $G_1 \boxtimes G_2$.

Let (u, v) be any vertex in $V - D$. Then u is adjacent to u_{d_l} for some l, where $1 \leq l \leq \lambda + 1$, because D_1 is a dominating set of G_1. Vertex v in G_2 is adjacent to any vertex $p_1, p_2, \ldots, p_k$ say p_m as D_2 is a dominating set of G_2. Hence by the definition of strong product, every vertex (u, v) of $V - D$ is adjacent to atleast one vertex (u_{d_l}, p_m) in D.

Thus D becomes a dominating set of $G_1 \boxtimes G_2$.

Suppose we delete a vertex (u_{d_w}, p_v) from D. Let vertex u_{d_w} be adjacent to the vertices $u_1, u_2, u_3, \ldots, u_{\varphi(n)}$ as degree of each vertex in G_1 is $\varphi(n)$. Here all the vertices $u_1, u_2, u_3, \ldots, u_{\varphi(n)}$ are not adjacent to the vertices of $D_1 - \{u_{d_w}\}$. If this is happened, then $D_1 - \{u_{d_w}\}$ becomes a dominating set of G_1, a contradiction to the minimality of D_1.

Therefore there is at least one vertex in $\{u_1, u_2, \ldots, u_{\varphi(n)}\}$, say u_r, which is not adjacent to any vertex of $D_1 - \{u_{d_w}\}$.

Since $p_1, p_2, \ldots, p_k$ are distinct primes, there is no adjacency between these vertices. Then by the definition of strong product, vertex (u_r, p_v) is not adjacent to any vertex of $D - \{(u_{d_w}, p_v)\}$. Hence $D - \{(u_{d_w}, p_v)\}$ is not a dominating set of $G_1 \boxtimes G_2$.

Thus D becomes a dominating set of $G_1 \boxtimes G_2$ with minimum cardinality $(\lambda + 1).k$.

Now we show that D is a matching dominating set of $G_1 \boxtimes G_2$. Two cases arise.

Case (i) : Suppose λ is odd. The vertices $u_{d_i}, u_{d_{i+1}}$ are adjacent since they are consecutive vertices in G_1. Hence by the definition of strong product, the vertices $(u_{d_i}, p_i), (u_{d_{i+1}}, p_i)$ are adjacent for $i = 1, 3, \ldots, \lambda$ in $G_1 \boxtimes G_2$. Further the edges in

$F = \{(u_{d_i}, p_1), (u_{d_{i+1}}, p_1); \ldots\ldots; (u_{d_i}, p_k), (u_{d_{i+1}}, p_k)\}$ for $i = 1, 3, \ldots, \lambda$

are pairwise non - adjacent, since there is no edge between the primes $p_1, p_2, \ldots, p_k$. Hence F forms a matching in $G_1 \boxtimes G_2$.

Now every vertex in $G_1 \boxtimes G_2$ is incident with some edge in F, as D is a dominating set. Since λ is odd there are an even number of edges in the above set. Thus $< F >$ admits a perfect matching and hence $< D >$ becomes a matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 1).k$.

Hence $\gamma_m(G_1 \boxtimes G_2) = (\lambda + 1)k$.

Case (ii): Suppose λ is even. Then $|F|$ is an odd number. So we cannot get a perfect matching in this case. Hence to get a matching dominating set in $G_1 \boxtimes G_2$, we include a vertex $u_{d_{\lambda+2}}$ to the dominating set D_1 of G_1 such that $u_{d_{\lambda+1}}$ and $u_{d_{\lambda+2}}$ are adjacent in G_1.

Let $D_1' = D_1 \cup \{u_{d_{\lambda+2}}\} = \{u_{d_1}, \ldots, u_{d_{\lambda+2}}\}$. Obviously D_1' is a dominating set of G_1 as D_1 is a dominating set of G_1.

Let $D' = D_1' \times D_2 = \{u_{d_1}, \ldots, u_{d_{\lambda+2}}\} \times \{p_1, p_2, \ldots, p_k\}$.

Obviously D' is a dominating set of $G_1 \boxtimes G_2$ as D is a dominating set of $G_1 \boxtimes G_2$. By using Case (i), we see that the edges in

$F = \{(u_{d_i}, p_1), (u_{d_{i+1}}, p_1); \ldots\ldots; (u_{d_i}, p_k), (u_{d_{i+1}}, p_k)\}$ for $i = 1, 3, \ldots, \lambda + 1$ are pair wise non - adjacent. Hence this set of edges form a matching in $G_1 \boxtimes G_2$. Now every vertex in $G_1 \boxtimes G_2$ is incident with some edge in F, as D' is a dominating set.

Since λ is even there are an even number of edges in the above set F. Thus $< F >$ admits a perfect matching and hence $< D' >$ becomes a matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 2).k$.

Now we claim that D' is minimum. Suppose we delete a vertex (u_{d_i}, p_k) from D'. Then $|D'|$ is an odd number which contradicts that D' is a matching dominating set.

Therefore D' is a minimum matching dominating set of $G_1 \boxtimes G_2$.

Hence $\gamma_m(G_1 \boxtimes G_2) = (\lambda + 2).k$.

Case 2: Suppose $\alpha_i = 1$ for more than one i.

We know that $D = \{u_{d_1}, \ldots, u_{d_{\lambda+1}}\}$ is a dominating set of G_1 and By Case 1 in Theorem 3.6.2 and Case 1 in Theorem 3.6.5 of Chapter 3 [6], we know that

$D_2 = \{p_1, p_2, \ldots, p_{i-2}, p_{i-1} \cdot p_i, p_{i+1} \ldots\ldots, p_k\}$ is a dominating set of G_2 with cardinality k.

Let

$D = D_1 \times D_2 =$
$\{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \times \{p_1, p_2, \ldots, p_{i-2}, p_{i-1} \cdot p_i, p_{i+1}, \ldots, p_k\}$.

Now we show that D is a minimum dominating set of $G_1 \boxtimes G_2$.

Let (u, v) be any vertex in $V - D$. Then u is adjacent to u_{d_l} for some l, where $1 \leq l \leq \lambda + 1$.

The vertex v in G_2 is adjacent tosome vertex in D_2, say p_m. Hence by the definition of strong product, every vertex (u, v) of $V - D$ is adjacent to atleast one vertex (u_{d_i}, p_m) in D. Thus D becomes a dominating set of $G_1 \boxtimes G_2$.

Now we show that D is minimal. Suppose we delete a vertex, say(u_{d_i}, p_j), $j = 1,2,\dots, i-2, i+1, \dots, k$ from D. Let vertex u_{d_i} be adjacent to the vertices $u_1, u_2, u_3, \dots, u_{\varphi(n)}$ as degree of each vertex in G_1 is $\varphi(n)$. Here all the vertices $u_1, u_2, u_3, \dots, u_{\varphi(n)}$ are not adjacent to the vertices of $D_1 - \{u_{d_i}\}$. If this is happened, then $D_1 - \{u_{d_i}\}$ becomes a dominating set of G_1, a contradiction to the minimality of D_1.

Therefore there is at least one vertex in $\{u_1, u_2, \dots, u_{\varphi(n)}\}$, say u_r, which is not adjacent to any vertex of $D_1 - \{u_{d_i}\}$.

Since $p_1, p_2, \dots, p_{i-2}, p_{i+1}, \dots, p_k$ are distinct primes, there is no adjacency between these vertices. Then by the definition of strong product, vertex (u_r, p_j) is not adjacent to any vertex of $D - \{(u_{d_i}, p_j)\}$. Hence $D - \{(u_{d_i}, p_j)\}$ is not a dominating set of $G_1 \boxtimes G_2$. Similar is the case with the deletion of any other vertex in D.

Thus D becomes a minimal dominating set $G_1 \boxtimes G_2$ with cardinality $(\lambda + 1).(k-1)$.

Now we show that D is a matching dominating set of $G_1 \boxtimes G_2$.Two cases arise.

Case (i) : Suppose λ is odd. The vertices $u_{d_i}, u_{d_{i+1}}$ are adjacent since they are consecutive vertices in G_1. Hence by the definition of strong product, the vertices $(u_{d_i}, p_j), (u_{d_{i+1}}, p_j)$ are adjacent for $i = 1, 3, \dots, \lambda$ in $G_1 \boxtimes G_2$.

Consider the set F of edges

$\{(u_{d_i}, p_j), (u_{d_{i+1}}, p_j); \dots; (u_{d_i}, p_{i-1}.p_i), (u_{d_{i+1}}, p_{i-1}.p_i)\}$ for

$i = 1, 3, \dots, \lambda$ and $j = 1,2, \dots, i-2, i+1, \dots, k$. No two edges in F are adjacent,since there is no edge between the primes $p_1, p_2, \dots, p_{i-2}, p_{i+1}, \dots, p_k$ and $GCD(p_j, p_{i-1}.p_i) = 1$. Hence F forms a matching in $G_1 \boxtimes G_2$.

Now every vertex in $G_1 \boxtimes G_2$ is incident with some edge in F, as D is a dominating set. Since λ is odd, there are an even number of edges in F. Thus $< F >$ admits a perfect matching and hence $< D >$ becomes a matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 1).(k-1)$.

Hence $\gamma_m(G_1 \boxtimes G_2) = (\lambda + 1).(k-1)$.

Case (ii): Suppose λ is even. In similar lines to Case (ii) of Case1, we include a vertex $u_{d_{\lambda+2}}$ which is adjacent to $u_{d_{\lambda+1}}$ to the dominating set D_1 of G_1.

Let $D_1' = D_1 \cup \{u_{d_{\lambda+2}}\} = \{u_{d_1}, \dots, u_{d_{\lambda+2}}\}$ Obviously D_1' is a dominating set of G_1 as D_1 is a dominating set of G_1.

Let

$D' = D_1' \times D_2 = \{u_{d_1}, \dots, u_{d_{\lambda+2}}\} \times \{p_1, p_2, \dots, p_{i-2}, p_{i-1}.p_i, p_{i+1} \dots, p_k\}$

. Obviously D' is a dominating set of $G_1 \boxtimes G_2$ as D is a dominating set of $G_1 \boxtimes G_2$.

Let F be the set of edges.

$\{(u_{d_i}, p_j), (u_{d_{i+1}}, p_j); \dots; (u_{d_i}, p_{i-1}.p_i), (u_{d_{i+1}}, p_{i-1}.p_i)\}$ for

$i = 1, 3, \dots, \lambda + 1$ and $j = 1,2, \dots, i-2, i+1, \dots, k$. By using above Case (i), we see that the edges in F are non - adjacent. Hence this set of edges form a matching in $G_1 \boxtimes G_2$. Now every vertex in $G_1 \boxtimes G_2$ is incident with some edge in F, as D' is a dominating set.

Since λ is even there are an even number of edges in the above set F. Thus $< F >$ admits a perfect matching and hence $< D' >$ becomes a matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 2).(k-1)$.

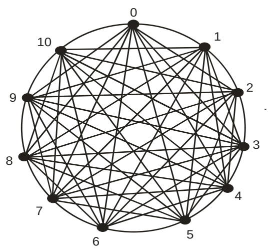
Now we claim that D' is minimum. Suppose we delete a vertex (u_{d_i}, p_k) from D'. Then $|D'|$ is an odd number which contradicts that D' is a matching dominating set.

Thus D' is minimal matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 2).(k-1)$.

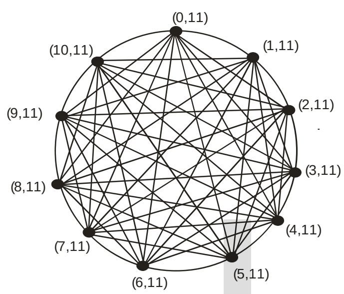
Hence $\gamma_m(G_1 \boxtimes G_2) = (\lambda + 2).(k-1)$.

5 GRAPHS

$n = 11$



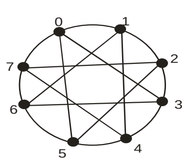
$G_1 = G(Z_{11}, \varphi)$ $G_2 = G(V_{11})$

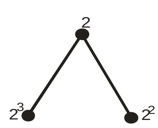


$G_1 \boxtimes G_2$

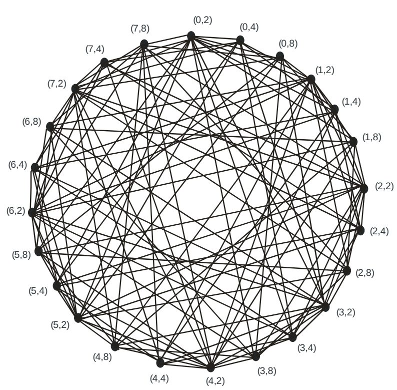
Matching dominating set {(0,11),(1,11)}

$n = 2^3 = 8$





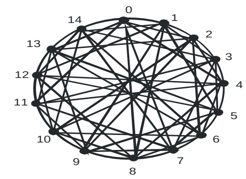
$G_1 = G(Z_8, \varphi)$ $G_2 = G(V_8)$

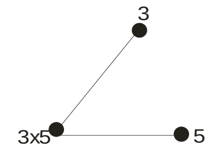


$G_1 \boxtimes G_2$

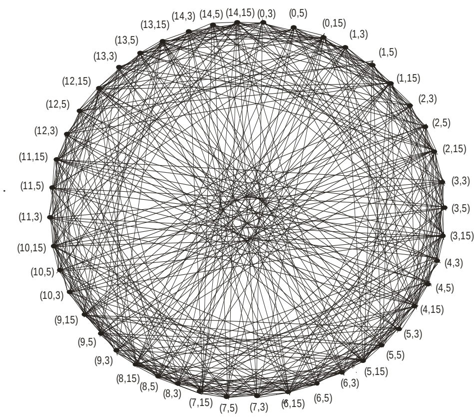
Matching dominating set $\{((0,2),(1,2))\}$

$n = 3 \times 5 = 15$





$G_1 = G(Z_{15}, \varphi)$ $G_2 = G(V_{15})$



$G_1 \boxtimes G_2$

Matching dominating set

$\{((0,15),(1,15)),((2,15),(3,15))\}$

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